SOME PROBLEMS OF HEAT CONDUCTION WITH PHASE TRANSITIONS

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An approximation of the problem of heating of a body by a surface heat flux of density 10^7-10^{14} W/m^2 is presented. Expressions are obtained for calculating the movement of the melting boundary and external destruction boundary and temperature conditions on the surface.

For engineering methods of calculating heat conduction processes with phase transitions it is necessary to know the solution of the nonlinear problem establishing the explicit functional relations between the initial parameters. In this case the use of approximate methods of solution is warranted.

An approximate method of solving problems of heat conduction with two phase transition boundaries is presented here. This class of problems is found, for example, when calculating conduction processes with consideration of melting and vaporization of metal under the effect of laser radiation, bombardment of an electrode by charged particles, and in other cases of the effect of intense heat flux sources.

The one-dimensional problem of heating of a body by a surface heat source with a moving external boundary of destruction and moving boundary of molten metal (Fig. 1) is described by the following differential equations and boundary conditions on the phase interface [1-3]:

$$\frac{\partial^2\vartheta_1}{\partial x^2}-\frac{1}{a_1}\cdot\frac{\partial\vartheta_1}{\partial t}=0,\quad X_0\leqslant x\leqslant X, \tag{1}$$

$$\frac{\partial^2\vartheta_2}{\partial x^2}-\frac{1}{a_2}\cdot\frac{\partial\vartheta_2}{\partial t}=0,\quad X\leqslant x, \tag{2}$$

$$\vartheta_1(X,\ t)=\vartheta_2(X,\ t), \tag{3}$$

$$\lambda_1\left(\frac{\partial\vartheta_1}{\partial x}\right)_{x=X}=\lambda_2\left(\frac{\partial\vartheta_2}{\partial x}\right)_{x=X}-L\rho\frac{dX}{dt}, \tag{4}$$

$$q+\lambda_1\left(\frac{\partial\vartheta_1}{\partial x}\right)_{x=X}=L_0\rho\frac{dX_0}{dt}, \tag{5}$$

$$\vartheta_2(x,\ 0)=V,\ \vartheta_2(\infty,\ t)=V.$$

To establish the relation between q(t) and the temperature of the moving external boundary of the body T_0 we use the Langmuir–Dushman equation [4]

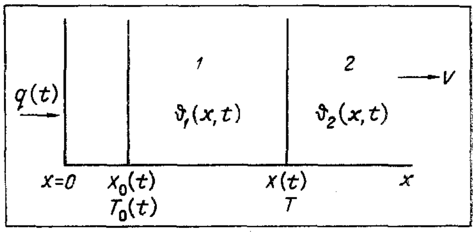

Fig. 1. Calculation model: 1) liquid phase; 2) solid phase.

$$\rho\frac{dX_0}{dt}=\sqrt{\frac{M}{T_0}}\exp 2.3026\left(A-4.234-\frac{B}{T_0}\right), \tag{6}$$

where A and B are tabulated constants.

With consideration of (6) the interrelation of q and T_0 is determined by the equation

$$q+\lambda_1\left(\frac{\partial\vartheta_1}{\partial x}\right)_{x=X}=L_0\sqrt{\frac{M}{T_0}}\exp 2.3026\left(A-4.234-\frac{B}{T_0}\right). \tag{7}$$

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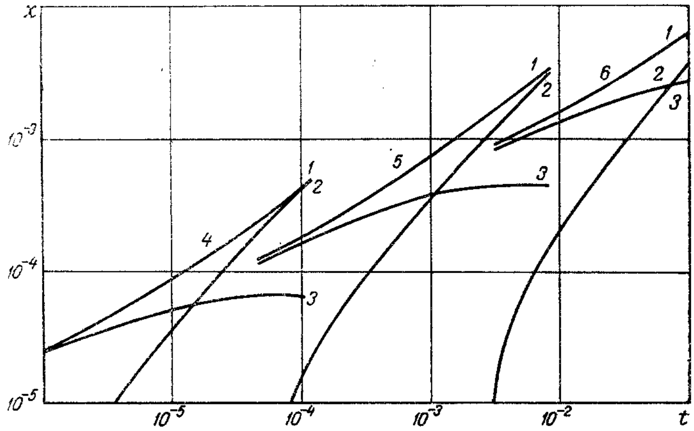

Fig. 2. Position of phase transition boundaries vs time for different values of the heat flux. Material is silver (x, m; t, sec): 1) X(t); 2) $X_0(t)$; 3) $(X-X_0)$; 4) 10^{11} W/m^2; 5) 10^{10} W/m^2; 6) 10^9 W/m^2.

Thus the problem reduces to a solution with boundary conditions of the second kind when the specific heat flux q is given.

On the basis of Leibenson's first method with consideration of the results of [5, 6] we assigned the temperature profiles of the liquid phase $\vartheta_1(x, t)$ and solid phase $\vartheta_2(x, t)$ in the form

$$\vartheta_1(x, t) = T_0 + \frac{T - T_0}{X - X_0}[x - X_0] + \frac{1}{2a_1} \cdot \frac{dT_0}{dt}[x - X_0][x - X], \tag{8}$$

$$\vartheta_2(x, t) = T - (T - V)\left\{1 - \exp\left[-\frac{1}{a_2} \cdot \frac{dX}{dt}(x - X)\right]\right\}. \tag{9}$$

Equations (8), (9) satisfy differential equations (1) and (2) at the phase interfaces and all boundary conditions enumerated above, with the exception of the conditions on the moving external boundary (5) and conditions (4) on the phase interface which are used for determining the remaining unknowns X_0 and X. The system of Eqs. (4), (5) reduces to a differential equation for the value of the fused zone $(X-X_0)$

$$y' = f(t)\,y^n + g(t)\,y + h(t), \tag{10}$$

where

$$y = X - X_0; \quad n = -1;$$

$$f(t) = \lambda_1 (T_0 - T)\left[\frac{1}{L_0\rho} + \frac{1}{L\rho + \frac{\lambda_2}{a_2}(T - V)}\right];$$

$$g(t) = \frac{\lambda_1}{2a_1} \cdot \frac{dT_0}{dt}\left[\frac{1}{L_0\rho} - \frac{1}{L\rho + \frac{\lambda_2}{a_2}(T - V)}\right];$$

$$h(t) = -\frac{q}{L_0\rho}.$$

The solution of the equation has the form [7]

$$y = \left(\frac{h}{f}\right)^{1/n} u(t), \tag{11}$$

where u(t) is determined by the relation

$$\int \frac{du}{u^n - \alpha u + 1} + C = \int \left(\frac{f}{h}\right)^{1/n} h\,dt.$$

The constant α is selected from the condition

$$\left(\frac{h}{f}\right)^{1/n} = e^{\int g\,dt}\left[\beta + \alpha \int h e^{-\int g\,dt}\,dt\right].$$

The law of motion of the moving external destruction boundary X_0 and law of motion of the liquid – metal interface X are determined then from differential equations (4), (5).

When q_0 = const and T_0 = const the problem reduces to the solution of the following differential equations:

$$\frac{dX}{dt}=\frac{\lambda_1(T_0-T)}{L\rho+\frac{\lambda_2}{a_2}(T-V)}\cdot\frac{1}{(X-X_0)}, \tag{12}$$

$$\frac{dX_0}{dt}=\frac{q_0}{L_0\rho}-\frac{\lambda_1(T_0-T)}{L_0\rho}\cdot\frac{1}{(X-X_0)}, \tag{13}$$

$$\frac{d(X-X_0)}{dt}=\lambda_1(T_0-T)\left[\frac{1}{L_0\rho}+\frac{1}{L\rho+\frac{\lambda_2}{a_2}(T-V)}\right]\frac{1}{(X-X_0)}-\frac{q_0}{L_0\rho}. \tag{14}$$

We introduce the notation of the complex which subsequently occurs often

$$\varepsilon=\lambda_1(T_0-T)\left[1+\frac{L_0}{L+c(T-V)}\right].$$

The solution of Eq. (14) is written in the form

$$X-X_0=-\frac{\varepsilon}{q_0}u(t), \tag{15}$$

where the function u(t) is determined from the expression for $u < 0$

$$u-\ln(u+1)=\frac{q_0^2t}{L_0\rho\varepsilon}. \tag{16}$$

For further calculations it is desirable to transform transcendental equation (16) to an explicit form in dimensionless time $t^* = q_0^2t/L_0\rho\varepsilon$.

Using the expansion of function $\ln(u + 1)$ in a series for the region of convergence of the series $-1 < u \leq 1$, we obtain

$$u(t)=-q_0\sqrt{\frac{2t}{L_0\rho\varepsilon}}\quad\text{for}\quad t^*\leqslant 0.1 \tag{17}$$

and

$$u(t)=-\frac{1}{2}\left[2\cos\frac{1}{3}\arccos\left(\frac{12\,q_0^2t}{L_0\rho\varepsilon}-1\right)-1\right]\quad\text{for}\quad t^*\leqslant 1/6. \tag{18}$$

For $t^* \geq 1/6$ the unknown function is approximated by the equation

$$u(t)=-1+0.5\exp\left[-\frac{3}{2}\left(\frac{q_0^2t}{L_0\rho\varepsilon}-\frac{1}{6}\right)\right]. \tag{19}$$

Using Eqs. (17)-(19), we determine the expressions for the value of the fused zone of metal and law of motion of the external destruction boundary and boundary of melting of the metal.

For small values of current time $0 \leq t^* \leq 0.1$

$$X-X_0=\sqrt{\frac{2\varepsilon t}{L_0\rho}}=\sqrt{\frac{2\lambda_1(T_0-T)}{L_0\rho}\left[1+\frac{L_0}{L+c(T-V)}\right]t}, \tag{20}$$

$$X=\sqrt{\frac{2\lambda_1(T_0-T)}{\rho[L+c(T-V)]}\left[\frac{L_0}{L_0+L+c(T-V)}\right]t}, \tag{21}$$

$$X_0=\frac{q_0t}{L_0\rho}-\sqrt{\frac{2\lambda_1(T_0-T)[L+c(T-V)]t}{L_0\rho[L_0+L+c(T-V)]}}. \tag{22}$$

For time $t^* \leq 1/6$

$$X-X_0=\frac{\varepsilon\varkappa}{2q_0}, \tag{23}$$

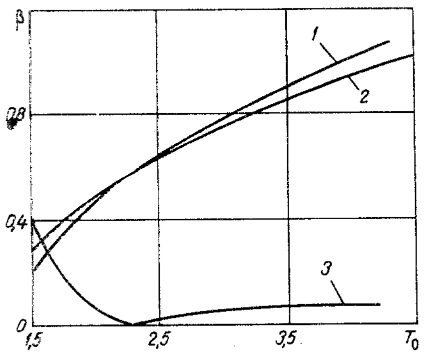

Fig. 3. Comparison of the exact and approximate solutions for one front of the phase transition (T_0, °K): 1) Neumann solution; 2) approximate solution; 3) relative error.

$$X = \frac{\lambda_1 (T_0 - T)}{2q_0} \cdot \frac{L_0}{[L + c(T - V)]} \left(\varkappa + \frac{1}{4}\varkappa^2\right), \tag{24}$$

$$X_0 = \frac{q_0 t}{L_0 \rho} - \frac{\lambda_1 (T_0 - T)}{2q_0} \left(\varkappa + \frac{1}{4}\varkappa^2\right), \tag{25}$$

where

$$\varkappa = 2 \cos \frac{1}{3} \arccos (12t^* - 1) - 1.$$

For a duration of the process $t^* \geq 1/6$

$$X - X_0 = \frac{\varepsilon}{q_0} \left\{1 - 0.5 \exp\left[-1.5\left(\frac{q_0^2 t}{L_0 \rho \varepsilon} - \frac{1}{6}\right)\right]\right\}, \tag{26}$$

$$X = \frac{\lambda_1 (T_0 - T)}{2q_0} \cdot \frac{L_0}{[L + c(T - V)]} \times \left\{1.84 + \frac{2q_0^2 t}{L_0 \rho \varepsilon} + \frac{4}{3} \ln \left(1 - 0.5 \exp\left[-1.5\left(\frac{q_0^2 t}{L_0 \rho \varepsilon} - \frac{1}{6}\right)\right]\right)\right\}, \tag{27}$$

$$X_0 = \frac{q_0 t}{L_0 \rho} - \frac{\lambda_1 (T_0 - T)}{2q_0} \left\{1.84 + \frac{2q_0^2 t}{L_0 \rho \varepsilon} + \frac{4}{3} \ln \left(1 - 0.5 \exp\left[-1.5\left(\frac{q_0^2 t}{L_0 \rho \varepsilon} - \frac{1}{6}\right)\right]\right)\right\}. \tag{28}$$

Analysis of Solution. The characteristic relations of the depth of the fused zone, position of the external destruction boundary, and value of fused zone are presented in Fig. 2. Heating of the body by a surface heat-flux source is accompanied in the initial stage only by melting of the metal. The duration of the transitional period to the start of vaporization from the surface is determined from Eq. (22)

$$t_0 = \frac{2\lambda_1 (T_0 - T) L_0 \rho}{q_0^2 \left[1 + \frac{L_0}{L + c(T - V)}\right]}, \tag{29}$$

or in relative units

$$t_0^* = \frac{2}{\left[1 + \frac{L_0}{L + c(T - V)}\right]^2}.$$

The surface temperature at the start of vaporization is found from Eq. (7)

$$q_0 = 2L_0 \sqrt{\frac{M}{T_0}} \exp 2.3026 \left(A - 4.234 - \frac{B}{T_0}\right). \tag{30}$$

The relation between the surface temperature and the density of the acting heat flux in a steady regime, when the fused zone of metal reaches the limiting value $(X - X_0)_{\lim} = \varepsilon/q_0$, is equal to

$$q_0 = [L_0 + L + c(T - V)] \sqrt{\frac{M}{T_0}} \exp 2.3026 \left(A - 4.234 - \frac{B}{T_0}\right). \tag{31}$$

We see from a comparison of Eqs. (30) and (31) that the surface temperature at the start of vaporization differs insignificantly from the temperature of the steady regime. The divergence does not exceed 10%, and therefore the process of destruction of the material under the effect of an intense heat source constant in time can be calculated on the assumption of constancy of the surface temperature.

The fused zone of metal reaches a steady value $(X - X_0)_{st} = 0.95\ (X - X_0)_{\lim}$ at time $t^* = 2$. The value of the fused zone is inversely proportional to the heat-flux density. The molten metal exists in all stages of the process, and vaporization occurs from the surface of the molten metal.

In the case of a duration of the effect of the heat flux $t^* > 2$ the rates of motion of the boundary of the molten metal and external destruction boundary are equal. The depth of the fused zone is determined by the energy of the heat flux and sum of the increment of enthalpy during transitions:

$$X = \frac{q_0 t}{\rho [L_0 + L + c(T - V)]}. \tag{32}$$

TABLE 1. Calculated and Experimental Values of the Depth of the Fused Zone

Material	q_0, W/cm^2	t, sec	X, mm	
			experiment	calculation
Duralumin	$1{,}5\cdot10^{10}$	$3\cdot10^{-7}$	2	1,67
Iron	$4{,}2\cdot10^{10}$	$1{,}2\cdot10^{-7}$	0,7	0,85
Iron	$2{,}3\cdot10^{6}$	$1{,}5\cdot10^{-3}$	0,6	0,6

TABLE 2. Comparison of Calculated and Experimental Values of T_0

Material	$t\cdot10^{-3}$, sec	s, cm^2	q_0, W/cm^2	T_0, °K		$\bar{\Delta}T=\frac{T_{calc}-T_{exp}}{T_{exp}}\times100\%$
				experiment	calculated	
Steel U-8	1,5	10^{-2}	$2{,}4\cdot10^{7}$	6400	5400	15,6
KhVG	0,5	$3\cdot10^{-4}$	$2{,}3\cdot10^{8}$	9600	7500	20,8
Armco iron	1,5	$6\cdot10^{-3}$	$2{,}3\cdot10^{6}$	4500	4260	6,6

For time $t < t_0$ there is no evaporation of the material from the surface of the body, and the approximate solution obtained can be compared with the exact Neumann solution for the case of melting of a body [1].

The law of motion of the liquid–metal interface X of the approximate and exact solution has the same functional relation

$$X = 2\beta (a_1 t)^{1/2},$$

where

$$\beta = \sqrt{\frac{\frac{\lambda_1}{a_1}(T_0 - T)}{2\rho\left[L + c(T - V)\right]}\left[\frac{L_0}{L_0 + L + c(T - V)}\right]}.$$

From the Neumann solution the constant of proportionality β_N is determined from the transcendental equation [1].

The values of the constants of proportionality β_N and β were calculated for W, Mo, Cu, Ag, and Al. Figure 3 gives the relations of β_N and β for silver and the values of the relative error

$$\bar{\Delta}\beta = \frac{\beta_N - \beta}{\beta_N}\, 100\%.$$

On the basis of the calculations we obtained that, for the metals considered with a wide range of variation of the coefficient of heat conductivity and boiling point, the error of the approximate solution does not exceed 10% in the range of heat fluxes $q_0 \geq 10^7$ W/m^2.

Tables 1 and 2 present the calculated values of the depth of melting of the body and surface temperatures and the experimental results of investigating the effect of laser pulses on metal [8, 9]. As we see from the table, the experimental values agree satisfactorily with the approximate solution.

Thus an analysis of the approximate solution of the problem of heating a body in the presence of phase transitions by a powerful heat-flux source showed that the relations obtained correctly describe the character of the processes and give a satisfactory accuracy when calculating the movement of the liquid–solid interface and destruction interface and temperature regimes on the surface.

NOTATION

q	is the heat flux;
$X_0(t)$	is the external destruction boundary;
$X(t)$	is the melting boundary;

T_0	is the temperature of destruction surface;
T	is the temperature at melting boundary, equal to the melting point;
V	is the initial temperature of body;
x	is the linear coordinate;
t	is the time;
λ	is the heat conductivity;
a	is the thermal diffusivity;
c	is the average specific heat of solid phase;
L_0, L	is the latent heat of vaporization and fusion;
ρ	is the density of material of the body.

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